

# **SYMMETRY of the RIEMANN OPERATOR**

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## **ABSTRACT**

Chaos quantization conditions, which relate the eigenvalues of a Hermitian operator (the Riemann operator) with the non-trivial zeros of the Riemann zeta function are considered, and their geometrical interpretation is discussed.

For a long time a challenge for mathematical physics has been and is still the idea, due to Hilbert at the dawn of the quantum age, to relate the non-trivial zeros of the Riemann zeta function with a spectrum of a Hermitian operator in a Hilbert space.

The Riemann zeta function  $\zeta(s)$  is defined for complex  $s = \sigma + i\rho$  and  $\text{Re } s > 1$  by the equation

$$\zeta(s) = \sum_1^{\infty} \frac{1}{n^s}. \quad (1)$$

The Riemann hypothesis [1] states that the non-trivial complex zeros of  $\zeta(s)$  lie on the critical line  $\text{Re } s = 1/2$ .

This is a very good motivation for understanding the original Riemann conjecture to find a self-adjoint operator  $\partial$  (or a positive one  $\partial(1-\partial)$ ) defined in a Hilbert space and with a spectrum given by the non-trivial zeros of the zeta function. In favour of this idea are the rigorous results [2], [3] on the spectral theory of the Laplace - Beltrami operator on the Poincaré complex upper half-plane, showing that the eigenfunctions are automorphic with respect to a discrete group of fraction-linear transformations. The most important application of representation theory to automorphic forms is the work of Selberg [4] who has shown that, if the fundamental domain of the discrete subgroup  $\Gamma \subset SL(2, R)$  is compact, then the spectrum of the elliptic operator for which the eigenvalue equation

$$-y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = \frac{1}{4}(1 + \rho^2)f \quad (2)$$

has  $\Gamma$ -invariant solutions is determined by the real values  $\rho_n$  given by  $\zeta_{\Gamma}(1/2 + i\rho/2) = 0$ , where  $\zeta_{\Gamma}(s)$  is a meromorphic function of  $s$  on the whole plane. The spectrum is continuous if the fundamental domain has a finite volume.

There has been theoretical [5], [13] and experimental [6], [7] evidence in support of the Riemann hypothesis, based on the fluctuations of spacing between consecutive zeros of zeta.

In a recent paper, Connes [8] has given a spectral interpretation of the critical zeros and a geometrical interpretation of the explicit formulas of number theory as a trace formula in a non-commutative space.

This work was inspired by the idea of Berry and Keating, discussed in Cambridge [9] and in a subsequent paper [10], that the real solutions  $E_n$  of

$$\zeta(1/2 + iE_n) = 0 \quad (3)$$

are energy levels, eigenvalues of a quantum Hermitian operator (the Riemann operator) associated with the one-dimensional classical hyperbolic Hamiltonian

$$H_d(x, p) = xp, \quad (4)$$

where  $x$  and  $p$  are the conjugate coordinate and momentum. They suggest a quantization condition generating Riemann zeros, for which however they ‘see no way to interpret geometrically’.

We propose that such a condition can be consistently obtained as a boundary quantization condition of a hyperbolic dynamical system within conformal geometry.

The characteristic properties of the classical dynamics (the Riemann dynamics) and the quantum analogue are listed and commented in [10]. It has the main features of a chaotic dynamics, namely instability, complex periodic orbits, no time-reversal symmetry, this in turn makes the quantum-classical correspondence not yet completely understood.

A prototype for the development of a theory of hyperbolic systems has been the free motion on a surface of constant negative curvature (a pseudosphere), which is one of the first models of chaotic motion. On this surface there exists a well-defined quantum dynamics where the Laplace - Beltrami operator acts as the Hamiltonian operator. The pseudosphere is modelled by the Poincaré unit disc or by the complex upper half-plane, which is the conformal image of the unit disc under a particular fraction-linear transformation [11]. The metric is conformal, proportional to the plane Euclidean metric at each point. This suggests an analysis of the hyperbolic dynamical system and its quantum analogue by applying the methods of conformal representation theory. A motivation for this is the geometry of points at infinity (or equivalently all directions along the cone), namely the property that any isometry of the hyperbolic space can be extended to a conformal diffeomorphism of the boundary sphere, which is known as the ‘Moebius transform’.

The classical Hamiltonian  $H = xp$  of linear dilation, i.e. multiplication in  $x$  and contraction in  $p$ , gives the Hamiltonian equations:

$$\dot{x} = x, \quad \dot{p} = -p. \quad (5)$$

Their solutions

$$x(t) = x_0 \exp(t), \quad p(t) = p_0 \exp(-t) \quad (6)$$

describe, for any  $E \neq 0$ , the classical trajectory of a hyperbola  $E = x_0 p_0$ . The system is unstable because it has a hyperbolic fixed point at  $x = 0, p = 0$ .

The system is quantized by considering the dilation operator in the  $x$  space

$$H = \frac{1}{2}(xp + px) = -i\hbar \left( x\partial_x + \frac{1}{2} \right), \quad (7)$$

which is the simplest formally Hermitian operator corresponding to the classical Hamiltonian (4).

The eigenvalue equation

$$H\psi_E(x) = E\psi_E(x) \quad (8)$$

is satisfied by the eigenfunctions

$$\psi_E(x) = \frac{C}{x^{1/2 - iE/\hbar}}, \quad (9)$$

where the complex constant  $C$  is arbitrary, since the solutions are not square-integrable.

The momentum eigenfunctions are found by the Fourier transform

$$\phi_E(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \psi_E(x) \exp(-ipx/\hbar) dx. \quad (10)$$

To calculate the integral the function  $\psi_E(x)$  has to be analytically continued across the singularity at  $x = 0$ . Following [10], one might simply choose even eigenfunctions, which is a natural choice, since the classical hyperbolic Hamiltonian (4) is parity-symmetric. The result is (by using the reflection and duplication formulas for the gamma function):

$$\begin{aligned} \phi_E(p) &= \frac{C}{|p|^{1/2+iE/\hbar} \sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp(-iu)}{|u|^{1/2-iE/\hbar}} du \\ &= \frac{C}{|p|^{1/2+iE/\hbar}} (h/\pi)^{iE/\hbar} \frac{\Gamma(1/4 + iE/2\hbar)}{\Gamma(1/4 - iE/2\hbar)}. \end{aligned} \quad (11)$$

Thus the quantum Hamiltonian (7) admits two independent stationary solutions [12] for any real energy  $E$

$$\psi_E(\pm x) = C\theta(\pm x)e^{(iE/\hbar - 1/2)\log|x|} \quad (12)$$

together with the corresponding momentum eigenfunctions. In (12)  $\theta(x)$  is the Heaviside step function. In [10] a more general relation between coordinate and momentum eigenfunction is considered; this is interpreted as two phase-space currents, the  $x$  current flowing out from the origin and the  $p$  current flowing into the origin.

The aim of our study is to obtain a discrete energy spectrum for the Hamiltonian (7) by imposing some quantization conditions. Quantization rules may arise in a natural way by the symmetries of the Hamiltonian (4). An obvious symmetry is the invariance under scale transformations:

$$x' = \lambda x, \quad p' = \frac{1}{\lambda} p \quad (13)$$

with  $\lambda$  positive. It follows from the classical equations of motion that the parameter of dilation  $\lambda$  corresponds to evolution after time  $T = \log \lambda$ . We shall consider the scale invariance of the hyperbolic dynamical system as a part of a larger group of transformations and discuss the Hamiltonian (4) from the point of view of conformal representations theory. Conformal symmetry in one-dimensional  $x$  space contains translations  $x' = x + a$ , dilations  $x' = \lambda x$ , and special conformal transformations. We note that the special conformal transformation

$$x' = \frac{x + cx^2}{1 + 2cx + c^2x^2} \quad (14)$$

can be written down as a superposition of two conformal inversions  $R$  (i.e. Weyl reflections),

$$Rx = -\frac{1}{x} \quad (15)$$

and a translation  $T_{-c}x = x - c$ ,

$$x' = RT_{-c}R. \quad (16)$$

These transformations are generated by the differential operators

$$\begin{aligned} P &= -i\partial_x, & D &= -id - ix\partial_x, \\ K &= -2idx - ix^2\partial_x, \end{aligned} \tag{17}$$

which form a realization of the Lie algebra of the conformal group in one dimension, isomorphic to the Lie algebra of  $SO(2,1) \sim SL(2,R)$ ;  $d$  is a complex number, a conformal (or scale) dimension that labels the irreducible representations (IR). We shall briefly review the conformal group in one dimension  $SO(2,1) \sim SL(2,R)$  and its principal series representation [14], which will be used in the following. The group  $SO(2,1)$  is the group of isometries of the pseudosphere

$$-y_0^2 + y_1^2 + y_2^2 = -R^2,$$

the image of which is the unit disc, by a stereographic projection onto the plane. The conformal group in one dimension, known as the real Moebius group  $SO(2,1) = SL(2,R)/Z_2$ , acts as a group of automorphisms of the compactification  $S^1$  of the real space  $\mathcal{R}$ , which can be realized as a set of ‘light’ rays belonging to the cone  $y_0^2 = y_1^2 + y_2^2$ . This set of rays is in a one-to-one correspondence with the unit circle in two dimensions. The  $SO(2,1)$  generators  $X_{ab} = -X_{ba}$ ,  $a, b = 0, 1, 2$  are expressed in terms of the conformal group generators by the relations:

$$P = X_{10} + X_{12}, \quad K = X_{12} - X_{10}, \quad D = X_{20}. \tag{18}$$

The image of the  $R$ -inversion (15) on the cone is a reflection of the axes  $y_1, y_2$ , i.e. space reflection. In fact the stability subgroup of the point  $x = 0$  (or  $y_1 = 0, y_0 = y_2 = \pm 1$  in homogeneous coordinates) is isomorphic to the Weyl subgroup. The latter is a subgroup of  $SL(2,R)$  of order 2 generated (*mod*  $\pm 1$ ) by a non-trivial element  $w$ , with  $w^2 = -1$ .

The unitary representations of  $G = SL(2,R)$  are induced by an unitary representation of its maximal compact subgroup  $K = SO(2)$  of rotations (generator  $X_{12}$ ). The representation theory is constructed in the space of functions with domain in the homogeneous space  $X = K \backslash G$  and values in a separable Hilbert space. The factor space  $X$  is isomorphic to the upper half-plane. It can be realized as the space of homogeneous functions of two variables of given parity and degree of homogeneity, determined by the conformal dimension  $-d$ . There is a one-to-one correspondence between the points of  $X$  and the points on the real line  $\mathcal{R}^1$ . An important class of unitary IR is the so-called principal series (and its dual) characterized by the value  $-d = -1/2 + i\rho$  (and  $-d = -1/2 - i\rho$  respectively), with  $\rho$  real. Let  $C^\infty$  be the space of infinitely differentiable (complex-valued) functions  $f(x)$  of given parity, which satisfy certain asymptotic conditions for  $|x| \rightarrow \infty$  following from the smoothness of  $f$  in the neighbourhood of the  $R$ -inversion element. The unitary (even) principal series IR acts in the Hilbert space completion  $\mathcal{H}^\mathcal{L}$  of  $C^\infty$  with respect to the scalar product

$$(f_1, f_2) = \int_{-\infty}^{\infty} \bar{f}_1(x) f_2(x) dx, \tag{19}$$

which is positive-definite and  $G$ -invariant. One can use any  $SO(2)$  invariant measure to construct a unitarily equivalent representation on the upper half-plane.

There exists a pair of intertwining maps relating the representation  $T^\chi, \chi = i\rho$  and the dual one  $T^{\tilde{\chi}}, \tilde{\chi} = -i\rho$ . It is defined by the convolution integral

$$(G_\chi f)(x_1) = \int_{-\infty}^{\infty} G_\chi(x_1 - x_2) f(x_2) dx_2, \quad (20)$$

and has the intertwining property

$$G_\chi T^{\tilde{\chi}} = T^\chi G_\chi.$$

The kernel of the integral operator (20) is given by

$$G_\chi(x) = \frac{N_\chi}{\sqrt{2\pi}} |x|^{-1+2i\rho}. \quad (21)$$

The Fourier transform of  $G_\chi$ ,

$$G_\chi(p) = \int_{-\infty}^{\infty} G_\chi(x) e^{-ipx} dx, \quad (22)$$

restricts the choice of the normalization constant  $N_\chi$  in such a way that, for equivalent representations, the product  $G_\chi G_{\tilde{\chi}}$  is proportional to the unit operator

$$G_\chi(p) G_{\tilde{\chi}}(p) = 1. \quad (23)$$

The Casimir operator of  $SO(2, 1)$

$$C_2 = -D^2 + \frac{1}{2}(PK + KP) = d(1 - d)$$

corresponds to the Laplace - Beltrami operator  $L$  [14], which is the second-order differential operator associated with the  $SL(2, R)$ -invariant metric on the upper-half plane  $z = x + iy$  and has the form  $L = -y^2 \Delta = -y^2(\partial_x^2 + \partial_y^2)$ . The solutions of the equation  $Lf = d(1 - d)f$  are invariant automorphic forms with respect to the discrete subgroups  $\Gamma$  of  $SL(2, R)$ . Automorphic forms are determined by the relations  $f(\gamma z) = f(z)$ , for  $\gamma \in \Gamma$ , which serve as boundary conditions for the operator  $L$ . They define the Laplace - Beltrami operator as a self-adjoint operator acting in a Hilbert space of square integrable functions with respect to the invariant measure in which a unitary representation of the group  $G$  is realized. The procedure of extending  $L$ , which is a positive, semi-bounded and symmetric operator with a dense domain, to a self-adjoint operator is known [14]. For unitary representations containing a stability vector with respect to the maximal compact subgroup, the spectral decomposition of  $L$  and the associated eigenfunctions - proper and generalized - are intimately connected with the discrete subgroup. According to the rigorous results on the  $L$ -theory if the discrete subgroup is such that the space  $G/\Gamma$  has a finite volume, the continuous spectrum entirely lies on the real half-axes  $1/4 \leq d(1 - d) \leq \infty$ , while the point spectrum lies on the line  $0 \leq d(1 - d) \leq \infty$ . The point spectrum is the set of singular points of the kernel of the resolvent of  $L$  for which the invariant bilinear functional in the representation space is degenerate. The automorphic eigenfunctions satisfying  $Lf = d(1 - d)f$  are generalized functions and form a complete set of functionals in the space of linear functionals on the unitary representation Hilbert space.

The coordinate eigenfunctions  $\psi_E(x)$  belong to the space (denoted hereafter  $\Omega^*$ ) of linear functionals on the even unitary principal series IR with  $\rho = E/\hbar$ . The position functions  $\psi_E(\frac{1}{x})$  also belong to the Hilbert space extension  $\Omega^* \supset \mathcal{H}^\mathcal{L}$ .

Returning to the eigenvalue equation (8), we may interpret it as a condition for a scale invariance of a conformal wave function of conformal dimension  $d = 1/2 - iE/\hbar$ . If we postulate invariance of the wave function  $\psi(x)$  under the full conformal group, this will give a *constant* eigenfunction. It can be easily verified that in addition to scale invariance the wave function  $\psi(x)$  is invariant under conformal  $R$ -inversion:

$$\psi(x) = \frac{1}{x^{2d}} \psi\left(-\frac{1}{x}\right). \quad (24)$$

Formula (24) defines the coordinate function as an automorphic form of weight  $d$  in the space of the IR principal series representation, invariant with respect to the discrete Weyl subgroup of  $SL(2, R)$ , where its non-trivial element is chosen to be the conformal  $R$ -inversion. (Automorphic is used here in the sense of invariant functions, associated with a stationary discrete subgroup of automorphisms of a manifold, which is the homogeneous space of a Lie group.)

The scale- and  $R$ -invariance properties of  $\psi(x)$  suggest that the quantum wave function of a chaotic dynamical system (4) is a scale-invariant function, i.e. a homogeneous function of degree  $-d$  and an  $R$ -inversion invariant automorphic function of weight  $d$ , where  $-d$  is the scale dimension that labels the unitary irreducible principal series representation  $T^\chi$  and is hence equal to  $-1/2 + iE/\hbar$ , with  $E$  real. The complex-conjugated wave function  $\bar{\psi}(x)$  corresponds to the dual representation  $T^{\bar{\chi}}$  with  $-d = -1/2 - iE/\hbar$ . The above suggestion means precisely that the wave function of such a chaotic system is related to the kernel (21) of the intertwining operator for the IR principal series and its dual. As such it belongs to the space  $\Omega^*$  of linear functionals on the unitary principal series IR Hilbert space. This assumption assures that the eigenvalues of the formally Hermitian Hamiltonian (7) are real.

We shall now show that the real values  $E$  are eigenvalues of a self-adjoint operator. For this purpose we consider the shifted Hamiltonian  $1/2 - iH/\hbar = -x\partial_x$ , which satisfies

$$x\partial_x(1 - x\partial_x) = -x^2\partial_x^2. \quad (25)$$

The operator on the RHS of (25) is the Laplace - Beltrami operator  $L$  on the real  $x$  line. It is readily seen that the functions

$$\psi_{\pm E}\left(\frac{1}{|x|}\right) = Cx^{1/2 \pm iE/\hbar}, \quad x > 0, \quad (26)$$

are eigenfunctions of this operator. We emphasize that the position eigenfunctions  $\psi(x)$  are even functions of  $x$ . As seen from the explicit expression (12) they are functions of the argument  $|x|$  only and can be considered as functions defined on the positive real line  $(0, \infty)$ . This assumption identifies  $\pm x$  and allows the use of a parametrisation such that the representation

space can also be realized as the Hilbert space of functions  $f(x)$ , defined on the half line  $x > 0$  and square-integrable with respect to the measure  $x^{-2}dx$ . It is known [14] that the functions

$$\theta(x, d) = x^{1/2+i\rho} + c(1/2 + i\rho)x^{1/2-i\rho}, \quad (27)$$

where  $c(d)c(1-d) = 1$ , with the boundary condition at  $a$  with an additional parameter  $\kappa$ ,

$$\theta(a) = a\kappa\theta'(a), \quad (28)$$

form a complete set of eigenfunctionals of the self-adjoint operator  $L$  in the space of linear functionals on the unitary principal series IR Hilbert space of functions  $f(x)$  defined on the positive line  $a \leq x \leq \infty, a \geq 1$  and square-integrable with respect to the measure  $x^{-2}dx$ .

Let  $\psi_{\pm E}(1/x)$  be the eigenfunctions of the Laplace - Beltrami operator  $-x^2\partial_x$  corresponding to the eigenvalues  $1/4 + E^2/\hbar^2$  of the continuous spectrum of  $L$ . It follows from (25) that

$$x\partial_x(1 - x\partial_x)x^{1/2 \pm iE/\hbar} = \left(\frac{1}{4} + \frac{E^2}{\hbar^2}\right)x^{1/2 \pm iE/\hbar}. \quad (29)$$

It is clear that  $x^{1/2 \pm iE/\hbar}$  are eigenvectors of the operators  $x\partial_x$  and  $1 - x\partial_x$ , corresponding to the eigenvalues  $1/2 \pm iE/\hbar$  and  $1/2 \mp iE/\hbar$  respectively. Thus if  $1/4 + E^2/\hbar^2$  belongs to the spectrum of  $L$  then  $1/2 \pm iE/\hbar$  belongs to the spectrum of the dilation operator  $x\partial_x$  and vice versa, so that roughly speaking the dilation operator is the square root of the Laplace - Beltrami operator in the Hilbert space of functions with domain  $(a, \infty)$  and square integrable with respect to the measure  $x^{-2}dx$ .

We turn now to the eigenvectors  $\psi(x)$ . As can be verified immediately they are eigenvectors of  $L$  but corresponding to complex eigenvalues  $1/4 - (1 \pm iE/\hbar)^2$ . However, there is an operator which, when acting on the position eigenfunctions  $\psi(x)$ , yields the real eigenvalues  $1/4 + E^2/\hbar^2$ . It is the  $R$ -transformed Laplace - Beltrami operator denoted hereafter  $L_R$ . It has the form

$$\begin{aligned} L_R &= -x\partial_x(1 + x\partial_x) \\ &= -x^2\partial_x^2 - 2x\partial_x. \end{aligned} \quad (30)$$

Expressing  $L_R$  in terms of the Hamiltonian (7):

$$L_R = (1/2 - iH/\hbar)(1/2 + iH/\hbar) \quad (31)$$

we obtain that the position functions  $\psi(x)$  are eigenfunctions of  $L_R$  corresponding to the eigenvalues  $1/4 + E^2/\hbar^2$ , so that the operators  $1/2 \pm iH/\hbar$  are formally the square roots of a positive operator. The elliptic Laplace - Beltrami operator and the  $R$ -transformed one act on functions whose arguments are related by the  $R$  transformation. The Laplace - Beltrami operator is densely defined and self-adjoint on the line  $[1, \infty)$ . The continuous  $R$ -transformation  $x' = 1/x$  on the line  $x > 0$ , maps its eigenfunctions to the eigenfunctions of the  $R$ -transformed operator  $L_R$ , which is hence densely defined and self-adjoint on the interval  $(0, 1]$ . The domains of the operators consist of elements satisfying the relation for automorphic  $R$ -invariant functions and



have as common elements the functions  $f$  defined for  $x = 1$ , which is the fixed point of the transformation  $x' = 1/x$ . The two operators have to be equal on the common elements. Relations (24), which determine the eigenfunctions  $\psi(x)$  as invariant automorphic functions with respect to the conformal  $R$ -inversion, serve as boundary conditions at the point  $x = 1$  (or at any point  $a \geq 1$ , if  $x' = a/x$ ), for coincidence of the inversed operator  $L_R = (1 - iH/\hbar)(1 + iH/\hbar)$  with the self-adjoint Laplace - Beltrami operator and will be used in phase space as quantization conditions that generate a discrete spectrum. Thus due to the  $\pm x$  identification the  $R$ -invariance automorphic relation defines the dilation Hamiltonian as a square-root of a self-adjoint operator. For the eigenfunctions of the Laplace - Beltrami operator, eq. (24) is equivalent to the Neumann boundary conditions for the functions and for their first derivatives:

$$\psi\left(\frac{1}{|x|}\right) = \psi\left(\frac{1}{|x|}\right), \quad \psi'\left(\frac{1}{|x|}\right) = -\psi'\left(\frac{1}{|x|}\right), \quad (32)$$

at the point  $|x| = 1$ . Clearly thanks to the additional parameter in (28) these conditions are trivially fulfilled. If not only the first of these conditions is automatically satisfied, while the second one has only the trivial solution, unless it is assumed that  $|x| = 1$  is a point where orientation is reversed. This seems to be a natural assumption, since conformal  $R$ -inversion is an orientation-reversing transformation, with fixed points  $|x| = 1$ , so that the points  $x$  may belong to manifolds of opposite orientation. Such are the curves of constant energy in phase space. The volume element in phase space is given by the area  $xp = h$  of the Plank cell with sides  $l_x$  and  $l_p$ . Contours of constant energy in phase space are the upper and lower branches of the hyperbola  $E = px$  and of the conjugated one  $E = -px$ , corresponding to the eigenvalues of the square root operator  $\pm H$  of the  $R$ -transformed Laplace - Beltrami operator. Being aware of the time non-invariance of the classical orbits [10], one should interpret the upper and lower branches of the conjugated hyperbola respectively as the combined space-time transform and the time transform of a given one. The  $R$ -inversions in the  $x$ -space are easily enlarged to transformations in phase space, which are area preserving and leave invariant the operators  $\pm H$ . These transformations have the form:

$$\begin{aligned} T_1^\pm : \quad x' &= -\frac{h}{x} & p' &= \pm \frac{x^2 p}{h}, \\ T_2^\pm : \quad x' &= -\frac{h}{p} & p' &= \mp \frac{xp^2}{h}. \end{aligned} \quad (33)$$

The upper  $\pm$ -indices denote that the determinant of the transformation is  $\pm 1$ . The  $-T_1$  transform of  $x$  is the conformal  $R$ -inversion corresponding to the value  $a = 2\pi$  in (28) (in units  $\hbar = 1$ ). For  $xp = h$  the transformations contain the canonical one  $x' = -p, p' = x$  and reflections  $x' = \pm x, p' = \pm p$ . Together with the transformations  $-T_1^\pm$  and  $-T_2^\pm$  (note that the identity is  $-T_2^+$ ), they generate a finite group which is the dihedral group  $\mathcal{D}_4$  of order 8 [15], consisting of all orthogonal transformations in two dimensions that preserve the regular 4-sided polygon centered at the origin. It is actually generated by reflections (denoted hereafter  $S_1$  and  $S_2$  respectively) about the  $x$  axis and about the (diagonal) line  $l$  inclined at an angle  $\pi/4$  to the positive  $x$ -axis. The rotations through multiples of  $2\pi/4$  form a cyclic subgroup of order 4. The invariant 4-gon is spanned by the vectors  $(\pm\sqrt{h}, 0), (0, \pm\sqrt{h}), (\pm\sqrt{h}, \pm\sqrt{h})$ . A fundamental region in the square is the 2-simplex with vertices  $(0, 0), (\sqrt{h}, 0), (\sqrt{h}, \sqrt{h})$ . The 2-simplices

are joint two by two to form four 2-complexes (the Plank cells in the distinct quadrants). The transformations fall into two classes depending on whether they preserve (reverse) sign of volume element, i.e. preserve (reverse) orientation. We can use the isometries that map one side of the fundamental region onto a side of an adjacent region to glue distinct regions together along their boundaries, imposing boundary and quantization conditions and forming a closed path corresponding to a classical closed orbit. By identifying all boundary points mapped to one another by an element of the group (i.e. points in each orbit of the group) we can impose as a boundary condition the requirement that the coordinate and the momentum eigenfunctions are automorphic invariant forms of the discrete subgroup of reflections:

$$\psi(x) = \psi(sx), \quad \phi(p) = \phi(sp), \quad (34)$$

for all  $s$  in the group. This is possible since for  $xp = \pm h$ , the  $x$  and  $p$  transformations each form a one dimensional factor map  $x' = T(x)$ ,  $p' = T(p)$  of the two dimensional map (33) of the 4-gon. The reflections in the  $x$ -factor map are the conformal  $R$ -transformation and the  $x$ -inversion and generate the Weyl subgroup of the dihedral group. The coordinate function  $\psi(x)$  is thus defined to be an invariant automorphic function with respect to the discrete  $R$ -transformation (which it is) and eq. (34) is equivalent to

$$\psi(x) = \frac{1}{x^{2d}}\psi(Rx), \quad \psi(x) = \psi\left(\pm\frac{h}{p}\right), \quad \phi(p) = \phi\left(\pm\frac{h}{x}\right). \quad (35)$$

Eqs. (35) impose relations between values of  $\psi$  and  $\phi$  at boundary points mapped into each other by an element  $s$ , generating the finite Weyl subgroup of the dihedral group. Since under the discrete  $R$ -transformation points transform to  $E > h$  with decreasing  $x$  and to  $E < -h$  with increasing  $x$  and taking the limit  $h \rightarrow 0$  yields the semiclassical approximation, it will be thus imposed as a boundary condition for a smooth transition from  $E < -h$  to  $E > h$ . An orientation is then assigned to each 2-cell and a direction to each edge, the natural direction, due to the defining homeomorphism to the unit interval  $[0, 1]$ , being from initial point 0 to final point 1. To each distinct element  $g_i, i = 1, 2, \dots, 8$  (i.e. root vector) of the group a vertex  $i$  is assigned and a directed edge  $(i, j)$  connects two vertices iff  $g_i = S_k g_j$  for a generator  $S_k$  of the group. A path extends from a given  $g$  at an edge of the fundamental simplex to  $S_{i_1} \dots S_{i_k} g$  and is closed if it can be presented in the form  $S_{i_1} \dots S_{i_k} = 1$ . This is a consequence of the relation  $(S_i S_j)^{m(i,j)} = 1$ , with  $m(i, i) = 1$ , for the Coxeter element [16] of the dihedral group, which is just the product of two generating reflections, hence is a rotation through  $2\pi/4$  of order 4. Denoting the rotation through  $\pi/2$  by  $r = S_1 S_2$ , we obtain that a closed path  $S_1 S_2 S_1 S_2$  is always of the form  $rr$ , which is the word representation of a projective plane. Thus the relation for the Coxeter element forces identification of antipodal boundary points and introduces topology of a projective plane [17] in phase space. At both  $l_x$  and  $l_p$  boundary sides of the 4-gon  $\pm x$  and  $\pm p$  are identified, the latter according to the equivalence relation  $(-\sqrt{h}, p) \simeq (\sqrt{h}, -p)$ , which endows the phase space with Moebius topology. It is readily seen that the Weyl reflection  $S_2$  and  $S_2^2$  form a cyclic subgroup of order 2, isomorphic to the group  $Z_2$ . With the identification of boundary points a closed path begins from a given (initial) point  $g_i = (x, p)$  and ends at a (final) antipodal one  $g_f = (-x, -p)$ . The closed paths  $\gamma_s$  fall into two homotopy classes depending on whether the final point is reached via a product of even or odd number of reflections  $S_2$ . In

the first case, the end point is connected to the initial one by a transformation with  $\det = 1$  and orientation is preserved along the path, while in the second case, the determinant of the transformation is  $-1$  and orientation is reversed along the path. Hence the two homotopy classes form a representation of the fundamental group of the projective plane  $\pi_1(\mathcal{RP}^2)$ , which is the non-trivial homology group  $Z_2$ . The boundary conditions for the automorphic functions have to be compatible with the topology of the projective plane, and in particular with the Moebius topology equivalence relation. Due to the twist the original 2-complexes (the Plank cells in the original quadrants) are mixed and eq. (35) becomes

$$\psi(x)|_{xp=h} = \frac{1}{x^{2d}} \psi\left(-\frac{1}{x}\right)|_{(-x)(-p)=h} = \frac{1}{x^{2d}} \psi\left(\frac{1}{x}\right)|_{x(-p)=h} \quad (36)$$

The boundary conditions in the form of eq. (36) are compatible with the identification of  $\pm x$ , and of  $\pm p$  by a twist, and serve at the same time as quantization conditions generating discrete spectrum. Indeed we have:

$$\begin{aligned} \psi(x) &= \frac{1}{x^{1-2iE/\hbar}} \frac{C}{x^{-1/2+iE/\hbar}} \\ &= \frac{1}{x} \frac{C}{x^{-1/2-iE/\hbar}}. \end{aligned} \quad (37)$$

Using further that  $x = h/\pm p$  and the expression (11) for the momentum eigenfunctions, we obtain

$$\psi(x) = \pm \frac{1}{x^{1/2}} p^{1/2} \phi(p) \pi^{iE/\hbar} \frac{\Gamma(1/4 - iE/2\hbar)}{\Gamma(1/4 + iE/2\hbar)}. \quad (38)$$

With the help of the functional equation for the Riemann zeta function

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s) \quad (39)$$

for  $s = 1/2 + iE/\hbar$ , we get from (38) the result

$$x^{1/2} \zeta(1/2 - iE/\hbar) \psi(x) = \pm p^{1/2} \zeta(1/2 + iE/\hbar) \phi(p), \quad x(\pm p) = h. \quad (40)$$

The latter formulas, with  $\pm x$  and  $\pm p$  identified, are consistent only if either  $\psi(x) = 0$  (and hence  $\phi(p) = 0$ ), or

$$\zeta(1/2 + iE/\hbar) = \zeta(1/2 - iE/\hbar) = 0, \quad (41)$$

which provides a regularization procedure for semiclassical approximation and generates a discrete real spectrum of the dilation Hamiltonian operator (7). Geometrically the conditions reflect the fact that the fundamental group of the projective plane is non-trivial, and can be mapped homomorphically to the automorphisms group of order two of the covering space. The defining homomorphism

$$\sigma : \pi_1(\mathcal{RP}^2, x) \rightarrow \pm 1 \simeq Z_2$$

assigns to each path  $\gamma_s$  a number  $\sigma(\gamma_s) = \pm 1$ , according to whether orientation is preserved or reversed by transport around it in such a way, that  $\sigma(\gamma_s) \neq 1$ , for  $\gamma_s \neq 1$ . This is consistent

with the group structure of the transformations (33), which form the double point dihedral group with the factorization property  $\mathcal{D}_4 = \mathcal{D}_2 \times Z_2$  (it can be presented equivalently as a semidirect product of the two cyclic groups  $Z_4 \cdot Z_2$ ) [18]. Hence the irreducible representations have a  $Z_2$ -grading too. They are twice as many, and given a representation  $\psi(x)$  of  $\mathcal{D}_2$  each element of the pair  $\sigma(\gamma_s)\psi(x)$  could be interpreted as associated with an orientation preserving or reversing homotopy class. For  $x$  belonging to a closed trajectory the quantization conditions (40) are an algebraic expression of the homomorphism  $\sigma$ , this unique homomorphism being the geometric representation of the finite Weyl (orthogonal) reflection group.

Employing the notation as in [10]

$$\langle x|\psi\rangle \equiv \psi(x), \quad \langle p|\psi\rangle \equiv \phi(p), \quad \hat{H} = H/\hbar, \quad (42)$$

we rewrite expression (40) in the form:

$$\begin{aligned} \langle x|\hat{x}^{1/2}\zeta(1/2 - i\hat{H})|\psi_E\rangle &= \pm \langle p|\hat{p}^{1/2}\zeta(1/2 + i\hat{H})|\psi_E\rangle, \\ xp = h, \quad (-\sqrt{h}, p) &\simeq (\sqrt{h}, -p). \end{aligned} \quad (43)$$

These relations can be geometrically interpreted as quantization conditions generating the Riemann zeros as a discrete spectrum of the energy operator of the conformally invariant Riemannian metric due to dihedral symmetry in phase space of the chaotic dynamical system. The Riemann zeros are the eigenvalues of the hyperbolic chaotic Hamiltonian for which the latter is the square root of the (inversed) Laplace - Beltrami operator.

The relation with the  $-$  sign on the RHS of (43) is exactly the quantization condition suggested by Berry and Keating in [10].

Remark: It is known that the surface symbol is not unique. Hence we can interpret the closed path either as a cylinder  $S_1S_2S_1^{-1}S_2$ , which is obtained from the 4-gon by identification of the  $l_x$ -sides only, or as a Moebius band  $S_1S_2S_1S_2$  with the  $l_p$ -sides identified by a twist. In the first case the boundary condition is the trivial one (the  $+$  sign in (43)) and is not a quantization condition. In the case of a Moebius band the boundary condition is the non-trivial one (the  $-$  sign in (43)) and serves as a quantization condition which generates Riemann zeros.

To summarize we have implemented conformal symmetry to model a chaotic dynamical system. The quantum Hamiltonian has a classical limit and a discrete real spectrum whose points are the Riemann zeros on the critical line  $\text{Res} = 1/2$ . The coordinate eigenfunctions are automorphic functions, invariant with respect to the discrete subgroup of Weyl reflections, for points on hyperbolic trajectories of constant energy in phase space. The smooth semiclassical behaviour around the saddle point in phase space is only possible for eigenvalues of the Hamiltonian, which are the critical zeroes of the Riemann zeta function.

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